Subsystem decreasing for exponential synchronization of chaotic systems

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Conditions are derived under which a general class of nonlinear dynamical systems admits chaotic synchronization. The result is applied to the Lorenz system, Rössler's second equation, a generalized Hopfield network, and a driven R-L-diode circuit. Several experimental as well as numerical results are also given to confirm the theory. [S1063-651X(99)13602-X]

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I. INTRODUCTION

The chaotic synchronization of Pecora and Carroll [1–4] has several interesting features, including potential applications. The scheme consists of two basic ideas—the appropriate decomposition of a nonlinear dynamical system into subsystems and the stability concept of the subsystems as generalized to chaotic systems. Chaotic synchronization is possible if each subsystem has negative conditional Lyapunov exponents. Theoretical results for conditional Lyapunov exponents are generally difficult to establish since Lyapunov exponents, by definition, involve existence arguments of various limits and the splitting of tangent spaces (of a vector field), as well as other properties.

This paper attempts to derive *a priori* conditions for a general class of nonlinear dynamical systems under which chaotic synchronization is achieved without checking the conditional Lyapunov exponents. The main concept involves *subsystem decreasing*. In the Pecora-Carroll scheme, a dynamical system is decomposed into two appropriate subsystems. The vector field of a subsystem may possess simple properties when the substate vector of the other substate vector is fixed, even though the whole (undecomposed) vector field is not simple. The primary result of this paper asserts that synchronization can be achieved if each of the decomposed subsystems satisfies a decreasing property (see Sec. II for a precise definition). The argument shows that if the subsystem decreasing is satisfied, then there is a properly performing Lyapunov function instead of Lyapunov exponents for the subsystems.

The result is applied to several nontrivial examples: the Lorenz system, Rössler's second equation, a generalized Hopfield network, and a driven R-L-diode circuit.

One difficulty associated with general Lyapunov function approaches [5] lies in the lack of a general synthesis method for Lyapunov functions. Our subsystem decreasing properties naturally lead to properly performing Lyapunov functions.

II. GENERAL RESULT

Consider a nonlinear dynamical system

$$\frac{dx}{dt}=F(x,y;u(t)),\quad \frac{dy}{dt}=G(x,y;u(t)), \tag{1}$$

where $x\in\mathbb{R}^n$, $y\in\mathbb{R}$, and $u(t)\in\mathbb{R}^r$. Associated with this system, consider

$$\frac{dx_s}{dt}=F(x_s,y;u(t)), \tag{2a}$$

$$\frac{dy_s}{dt}=G(x_s,y_s;u(t)). \tag{2b}$$

This formulation allows nonautonomous systems where $u(t)$ typically represents the driving force of the system. Although the two systems consist of the same functions F and G, one of the arguments y in Eq. (2a) is *not* y_s, and this difference also affects the second equation (2b). Note that the ''slave'' equations are nonautonomous even when $u(t)$ is empty, because it is driven by $y=y(t)$. For further clarification, we will define our use of synchronization in this paper.

Definition (exponential synchronization):

(1) The first slave system (2a) exponentially synchronizes with its master system [Eq. (1)] if

$$\|x(t)-x_s(t)\|\leqslant e^{-\gamma_1 t}k,\quad 0<k_1,\gamma_1,$$

where k_1 may depend on $(x(0),y(0),y_s(0))$, while γ_1 should not depend on initial conditions.

(2) The slave system (2a) together with Eq. (2b) exponentially synchronizes with Eqs. (1) if in addition to the exponential synchronization of Eq. (2a),

$$\|y(t)-y_s(t)\|\leqslant e^{-\gamma_2 t}k_2,\quad 0<k_2,\gamma_2,$$

holds.

Remarks:

(i) He and Vaidya [5] define the synchronization of two dynamical systems by demanding that $x_s(t)\to x(t)$ as $t\to\infty$ and demonstrates that this is equivalent to the asymptotic stability of Eq. (2a). There is no requirement for $y_s(t)$. As previously stated, Eq. (2a) is nonautonomous even when $u(t)$ is absent. The asymptotic stability of nonautonomous systems is strikingly different from that of autonomous systems and is difficult to check. To demonstrate this, consider the variational equation associated with Eq. (2a),

$$\frac{dz}{dt}=\frac{\partial F}{\partial x_s}(x_s(t),y(t);u(t))z, \tag{3}$$

and let $\{\lambda_i(t)\}_{i=1}^n$ be eigenvalues of $(\partial F/\partial x_s)(x_s(t), y(t);u(t))$. It is simply untrue that

$$\mathrm{Re}\,\lambda_i(t)<0, \quad i=1,...,n, \tag{4}$$

for all t implies asymptotic stability. An example is given in Appendix A where Eq. (4) does not imply asymptotic stability.

(ii) Note also that conditional Lyapunov exponents presume continuous invariant splitting of the tangent spaces of time-dependent vector fields, which is extremely if not impossibly difficult to check. Furthermore, Lyapunov exponents are, by definition, associated with long-term average properties, and so care must be taken in studying chaotic synchronization.

(iii) These observations naturally lead us to seek simple *a priori* conditions for synchronization that are valid for a reasonably large class of nonlinear dynamical systems. The following result gives a simple *a priori* test for exponential synchronization. We will demonstrate later that a reasonably large class of systems satisfies the conditions to be given. Our result comes from the fact that $F(x_s;y;u(t))$ with respect to x_s, *when* $(y,u(t))$ is fixed, can behave in a much simpler manner than $(F(x_s,y;u(t)),G(x_s,y_s;u(t)))$ with respect to (x_s,y_s) jointly. Similarly, $G(x_s,y_s;u(t))$ can have simple properties with respect to y_s when $(x_s,u(t))$ is fixed.

A demonstration of the following fact is given in Appendix B.

Proposition 2.1. Assume that the dynamical systems [Eqs. (1) and (2)] admit unique uniformly bounded solutions. First slave system (2a) exponentially synchronizes with master system [Eq. (1)] if (A) F is decreasing *with respect to x*; i.e., there is an $\alpha>0$ such that

$$(x-x')^T[F(x,y;u)-F(x',y;u)]\leqslant-\alpha\|x-x'\|^2, \tag{5}$$

where T denotes the vector transpose.

Slave system (2a) together with Eq. (2b) exponentially synchronizes with Eqs. (1) if (B) G is decreasing *with respect to y* and Lipschitz *with respect to x*; i.e., there are $\beta>0$ and $L>0$ with

$$(y-y')[G(x,y;u)-G(x,y';u)]\leqslant-\beta(y-y')^2, \tag{6}$$

$$|G(x,y;u)-G(x',y;u)|\leqslant L\|x-x'\|. \tag{7}$$

Remarks. Strictly speaking, the decreasing should be uniformly decreasing. Similarly, Lipschitz should be uniformly Lipschitz; however, we prefer simplicity to complexity.

III. APPLICATIONS

A reasonably large class of nonlinear circuits and systems satisfies conditions (A) and (B) given in the previous section. This section confirms the validity of the general results for exponential synchronization by applying them to specific nontrivial examples.

Section III A demonstrates that the exponential synchronization of the Lorenz system can be checked by proposition 2.1. Section III B examines Rössler's second equation and shows that the first slave system admits exponential synchronization. Section III C verifies that a generalized Hopfield network can admit exponential synchronization. Section III D discusses a driven R-L-diode circuit and gives a theoretical justification for exponential synchronization together with an experimental verification; this system is now one of very few real physical systems where chaotic exponential synchronization is experimentally observed and theoretically verified.

A. Lorenz system

If the first coordinate u of the Lorenz system,

$$\dot{u}=\sigma(v-u), \tag{8}$$

$$\dot{v}=\rho u-v-uw, \tag{9}$$

$$\dot{w}=-bw+uv, \tag{10}$$

drives the (v_s,w_s) subsystem,

$$\dot{v}_s=\rho u-v_s-uw_s, \tag{11}$$

$$\dot{w}_s=-bw_s+uv_s, \tag{12}$$

and the resulting v_s drives the second subsystem,

$$\dot{u}_s=\sigma(v_s-u_s), \tag{13}$$

then $y=u$, $x=(v,w)$, $x_s=(v_s,w_s)$, and $y_s=u_s$ in terms of Eqs. (1) and (2) in Sec. II. Since u in Eqs. (11) and (12) are *not* u_s, the right sides of Eqs. (11) and (12) are *jointly linear* with respect to (v_s,w_s), i.e., $F(x_s,y)$ in Eqs. (2) of Sec. II or, what amounts to the same, $F(x,y)$ in Eqs. (1) is linear with respect to x, and

$$\begin{aligned}(x-x')^T&[F(x,y)-F(x',y)]\\ &=-(v-v')^2-u(v-v')(w-w')\\ &\quad+u(v-v')(w-w')-b(w-w')^2\\ &=-[(v-v')^2+b(w-w')^2]\leqslant-\alpha\|x-x'\|^2,\end{aligned}$$

where

$$\alpha=\min(1,b),$$

which is positive if $b>0$; so F is decreasing in the sense of Eq. (5). Since Eq. (13) is linear with respect to u_s with coefficient $-\sigma$, the decreasingness of G [Eq. (6)] is trivially satisfied, provided that $\sigma>0$.

B. Rössler's second equation

Of the four prototype equations for chaos [6] Rössler's second equation

$$\begin{aligned}\dot{u}&=u-uv-w,\\ \dot{v}&=u^2-av,\\ \dot{w}&=b(u-w)\end{aligned} \tag{14}$$

exhibits ''inverted spiral-plus saddle-type'' chaos as shown in Fig. 1 where $a=0.1$, $b=0.08$, and $c=0.125$. Let $x:=(v,w)$ and $y:=u$ so that

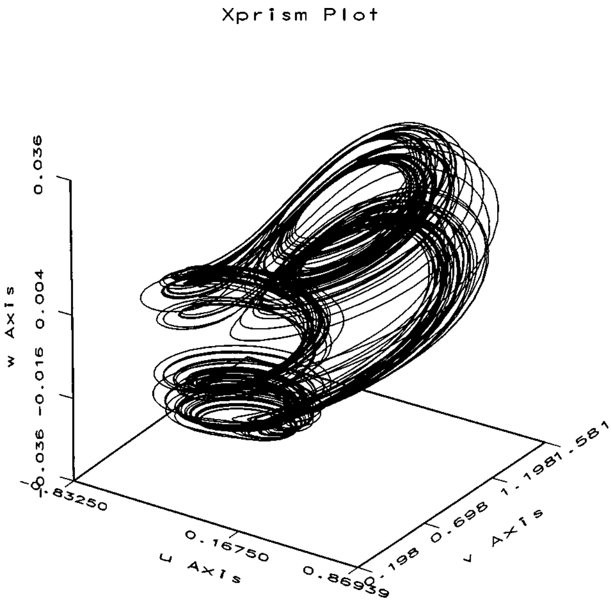

FIG. 1. Chaotic attractor from Rössler's second equation.

$$\dot{v}_s = -av_s + u^2,$$
$$\dot{w}_s = -bw_s + bcu. \tag{15}$$

For u fixed, each of the two equations in Eqs. (2) is linear with negative coefficient and hence condition (A) is satisfied. This guarantees the x-x_s synchronization as shown by Fig. 2. The y-y_s synchronization, on the other hand, is not guaranteed since

$$\dot{u}_s = -us(v_s - 1) - w_s \tag{16}$$

and since v_s becomes less than 1 from time to time.

C. Generalized Hopfield network

The well-known Hopfield network [7] is described by (Fig. 3)

$$C\frac{dv_i}{dt} = -\frac{1}{R}v_i + \sum_{j\neq i}^{N} T_{ij}\sigma(v_j) + I_i, \quad i = 1,...,N, \tag{17}$$

where C is the capacitance, R is the resistance, $\{T_{ij}\}$ represents couplings, σ is the (nonlinear) voltage characteristic, and I_i represents the current source injected into the ith sub-

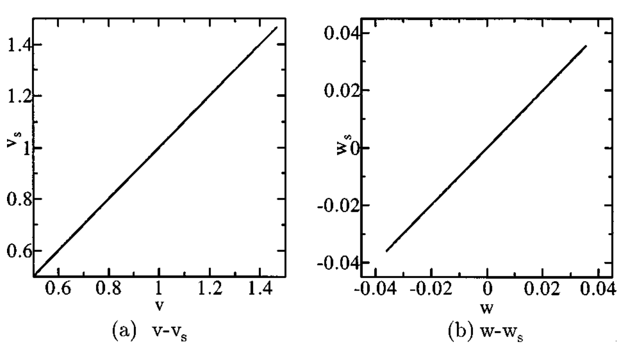

FIG. 2. Synchronization is achieved for (v, v_s) and (w, w_s).

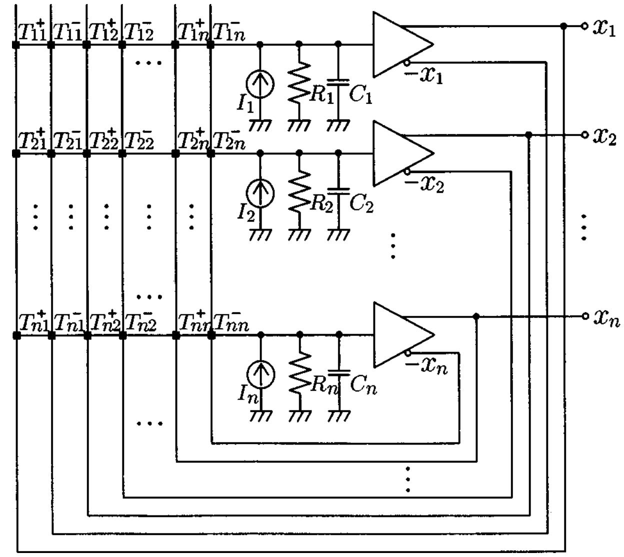

FIG. 3. Hopfield network. An arrow indicates an independent current source.

circuit. In its original form, the following are assumed for Hopfield networks: (i) $T_{ij} = T_{ji}$, (ii) σ is continuous and sigmoidal (i.e., it is strictly monotonically increasing and uniformly bounded), and (iii) I_i is constant.

We relax conditions (i) and (iii) so that $\{T_{ij}\}$ can be asymmetric and $I_i = I_i(t)$ can be time dependent. Consider the following decomposition of Eq. (17):

$$C\frac{dx_i}{dt} = -\frac{1}{R}x_i + \sum_{j\neq i}^{N-1} T_{ij}\sigma(x_j) + T_{iN}\sigma(y) + I_i(t), \tag{18}$$

$$i = 1,...,N-1,$$

$$C\frac{dy}{dt} = -\frac{1}{R}y + \sum_{j=1}^{N-1} T_{Nj}\sigma(x_j) + I_N(t), \tag{19}$$

where $y := x_N$.

Let $x := (x_1,\dots,x_{N-1})$ and let $\Sigma(x) := (\sigma(x_1),\dots,\sigma_{N-1}(x_{N-1}))$. If $-(1/R)x + \Sigma(x)$ is decreasing, then exponential synchronization is possible, because Eq. (19) is linear with respect to y and $\sigma(x_j)$ are Lipschitz. If $N=2$, then $x = x_1$, $y = x_2$, and $-(1/R)x + \Sigma(x) = -(1/R)x$; so no condition is required. Specifically, consider

$$\frac{dx}{dt} = -\frac{1}{R}x + \beta\sigma(y) + I\sin\omega t,$$
$$\frac{dy}{dt} = -\frac{1}{R}y - \beta\sigma(x) + I\sin\omega t, \tag{20}$$

$$\frac{dx_s}{dt} = -\frac{1}{R}x_s + \beta\sigma(y) + I\sin\omega t, \tag{21}$$

$$\frac{dy_s}{dt} = -\frac{1}{R}y_s - \beta\sigma(x_s) + I\sin\omega t. \tag{22}$$

Figure 4 shows a chaotic attractor of the (x,y) dynamics (master system) with

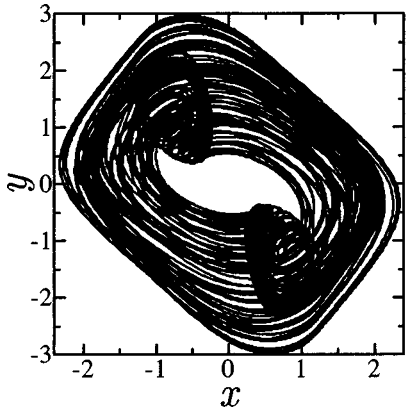

FIG. 4. Chaotic attractor from a generalized Hopfield network.

$$R = 30.7, \quad \beta = 0.615\,16, \quad I = 0.507\,92, \quad \omega = 0.131\,82.$$

Figures 5(a) and 5(b) show (x, x_s) and (y, y_s) plots, which indicate synchronization.

D. Driven *R*-*L*-diode circuit

1. Theory

Consider the driven R-L-diode circuit given by Fig. 6. The diode is accurately characterized by a parallel connection of three nonlinear elements: (i) nonlinear resistor, (ii) nonlinear diffusion capacitor C_d, and (iii) nonlinear junction capacitor C_j [8]. While the first two components are relatively easy to describe, the last component C_j is difficult, particularly in the positively biased region. We represent the combined characteristics of C_d and C_j by f without attempting to give analytical expressions. The dynamics of the circuit is then described by

$$\frac{dq}{dt} = i - g(f(q)), \tag{23}$$

$$L\frac{di}{dt} = -Ri - f(q) + e(t), \tag{24}$$

where q is the total charge stored in the two capacitors and g is the nonlinear resistor, a well-known exponential function.

The first result of the chaotic behavior of this system was reported by Linsay [9]. The rich variety of bifurcations, including the chaotic behavior of the R-L-diode circuit [10–31], comes from the nonlinearity of the capacitor in the diode

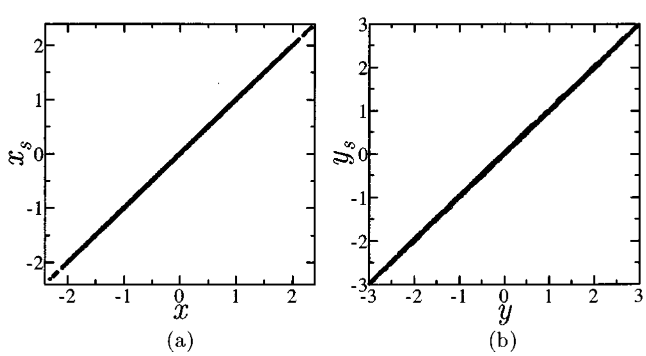

FIG. 5. Synchronization in the generalized Hopfield network.

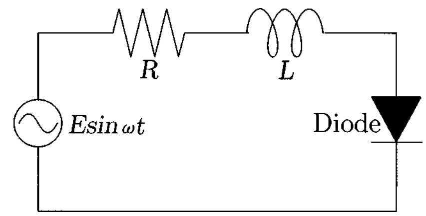

FIG. 6. An R-L-diode circuit driven by a sinusoidal voltage source.

[8]. Synchronization of this system using occasional proportional feedback is reported in [32]. The naturalness of the system is evidenced by the fact that very similar dynamics (equations) arise from different systems [33,34].

Corresponding to this master system, we consider the slave system driven by i of the master system:

$$\frac{dq_s}{dt} = i - g(f(q_s)), \tag{25}$$

$$L\frac{di_s}{dt} = -Ri_s - f(q_s) + e(t). \tag{26}$$

The following fact is shown in Appendix C by using proposition 2.1.

Proposition 3.1. The slave system described by Eqs. (25) and (26) exponentially synchronizes with its master system (23), (24) if $-f$ and $-g$ are decreasing, f is Lipschitz, and $e(t)$ is continuous and satisfies $|e(t)| \leqslant B$.

2. Experiment

When $R = 10[\Omega]$, $L = 50[\text{mH}]$, D:6CC13, $E_m = 4.0[\text{V}]$, and $f = 10[\text{kHz}]$, chaotic synchronization is observed (Figs. 7 and 8) where the master system is chaotic. Synchronization is not achieved when the slave diode is replaced by a different model while everything else remains the same (Fig. 9).

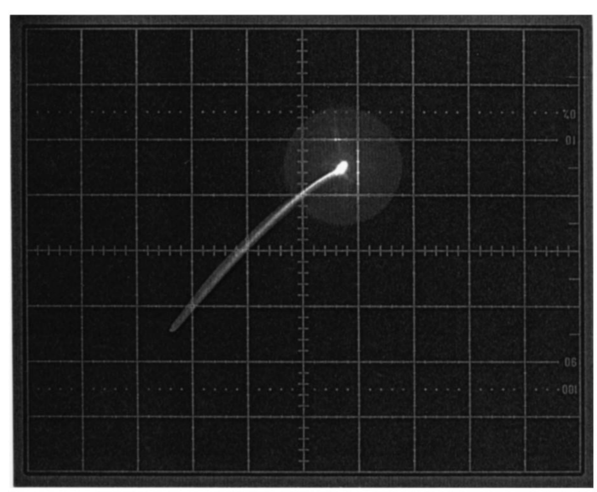

FIG. 7. Graph of $V_{d_{Slave}}$ vs $V_{d_{Master}}$ lies on the 45° diagonal: synchronization. Vertical and horizontal scale: 2V/div.

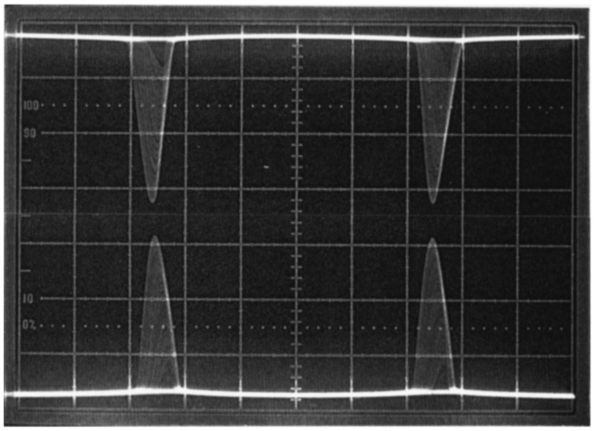

FIG. 8. Master diode voltage (top) and slave diode voltage (bottom, polarity inverted): synchronization. Vertical scale: 2V/div, horizontal scale: 20μs/div.

E. More general class of circuits

There is no dynamics in a resistive circuit; i.e., the dynamics of a circuit is induced by capacitors and inductors. A typical capacitor, shown in Fig. 10(a), is described by

$$\frac{dq}{dt} = i_c - i(v_c(q)),$$

where the resistor can be parasitic; q, i_c, and v_c are the charge, current, and voltage associated with the capacitor; and i is the resistor current. The dependences of i on v_c and v_c on q are often monotone, as is the case with the R-L-diode circuit. Similarly, a typical inductor, shown in Fig. 10(b), is described by

$$\frac{d\phi}{dt} = v_L - v(i_l(\phi)),$$

where ϕ, v_L, and i_L are the flux, voltage, and current associated with the inductor, and v is the resistor voltage. Again,

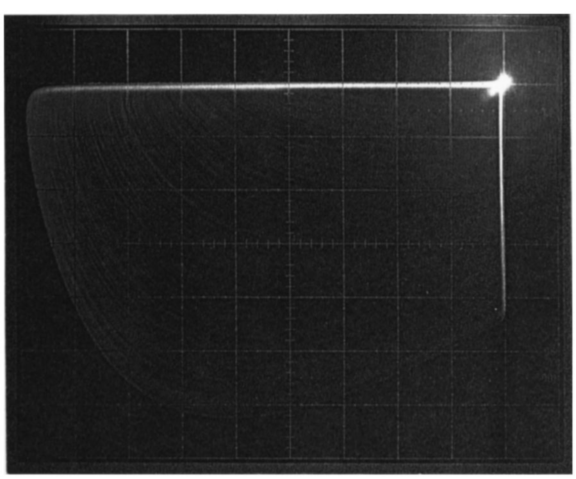

FIG. 9. Graph of $V_{d_{Slave}}$ vs $V_{d_{Master}}$ does not lie on the 45° diagonal when the slave diode is replaced by different type: asynchronization. Vertical and horizontal scale: 2V/div.

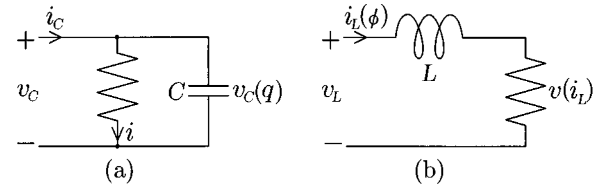

FIG. 10. General class of circuits.

v often depends on i_L in a monotonic manner and $i_L(\phi)$ is often monotone, as is the case with the driven R-L-diode circuit.

A general circuit is an interconnection of these elements together with resistors and transistors. There is a large class of circuits that satisfies the required decreasingness conditions. An elaborate description of this class of circuits is beyond the scope of this paper.

IV. CONCLUDING REMARKS

A general criterion is given of chaotic synchronization without reference to conditional Lyapunov exponents, the latter being difficult to check by definition. The criterion is used to show chaotic synchronization of several nontrivial systems. The required conditions for synchronization are often not difficult to check because the conditions require decreasing of a function with respect to a subvector instead of a full vector.

An interesting scheme associated with the chaotic synchronization is chaotic masking [35–37] where information signal is embedded into a (chaotic) drive signal and transmitted to receiver. A simple decoding system at the receiver can ''restore'' the original information. Our experimental results with an R-L-diode circuit show that the ''decoded'' information has amplitude which is much greater than the original which gives rise to a new open problem. The results should be reported elsewhere.

ACKNOWLEDGMENTS

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APPENDIX A

Consider [38]

$$\frac{dz}{dt} = A(t)z,$$

$$A(t) = \begin{bmatrix} -1+\frac{3}{2}\cos^2 t & 1-\frac{3}{2}\cos t \sin t \\ -1-\frac{3}{2}\cos t \sin t & -1+\frac{3}{2}\sin^2 t \end{bmatrix}. \quad \text{(A1)}$$

The eigenvalues of $A(t)$ are $\lambda_{1,2}(t) = (-1 \pm i\sqrt{7})/4$, which have *a negative real part uniformly* with respect to t, and yet one can show that

$$z(t)=e^{t/2}\begin{bmatrix}-\cos tz_1(0)\\ \sin tz_2(0)\end{bmatrix},\quad z_1(0)=z_2(0), \tag{A2}$$

is a solution for Eq. (A1), which obviously blows up as $t\to\infty$.

APPENDIX B

Since $x(t)$ and $x_s(t)$ are assumed to be uniformly bounded,

$$V_1(x,x_s):=\frac{1}{2}\|x-x_s\|^2$$

is well defined. It follows from the decreasing of F with respect to x that

$$\frac{dV_1(t)}{dt}=-(x-x_s)^T[F(x,y;u(t))-F(x',y;u(t))]$$
$$\leqslant-\alpha\|x-x_s\|^2=-2\alpha V_1(t), \tag{B1}$$

so that

$$V_1(t)\leqslant e^{-2\alpha t}V_1(0). \tag{B2}$$

This shows exponential synchronization of Eq. (2a) to Eqs. (1).

Next let

$$V_2:=|y-y_s|. \tag{B3}$$

Then

$$\frac{dV_2(t)}{dt}=G(x,y;u(t))-G(x_s,y_s;u(t))$$
$$=G(x,y;u(t))-G(x,y_s;u(t))+G(x,y_s;u(t))$$
$$-G(x_s,y_s;u(t)), \tag{B4}$$

provided that $y-y_s>0$. It follows from the decreasing of G with respect to y [see Eq. (5)] that

$$G(x,y;u(t))-G(x,y_s;u(t))\leqslant\beta(y-y_s),\quad y>y_s. \tag{B5}$$

That G being Lipschitz with respect to x [see Eq. (6)] implies

$$|G(x,y_s;u(t))-G(x_s,y_s;u(t))|\leqslant L\|x-x_s\|. \tag{B6}$$

Equations (B4)–(B6) give

$$\frac{dV_2(t)}{dt}\leqslant-\beta V_2(t)+L\sqrt{2V_1(0)}e^{-\alpha t},\quad y>y_s. \tag{B7}$$

An elementary fact in the differential inequality [39] gives

$$V_2(t)\leqslant e^{-\beta t}V_2(0)+L\sqrt{2V_1(0)}\int_0^t e^{-\beta(t-\tau)}e^{-\alpha\tau}d\tau$$
$$=e^{-\beta t}V_2(0)+\frac{L\sqrt{2V_1(0)}}{\beta-\alpha}(e^{-\alpha t}-e^{-\beta t}), \tag{B8}$$

provided that $y>y_s$ and $\beta\neq\alpha$. If, on the other hand, $y<y_s$, then $V_2=-(y-y_s)$ and

$$\frac{dV_2(t)}{dt}=-[G(x,y;u(t))-G(x,y_s;u(t))]$$
$$-[G(x,y_s;u(t))-G(x_s,y_s;u(t))]$$
$$\leqslant\beta(y-y_s)+L\sqrt{2V_1(t)}$$
$$=-\beta V_2(t)+L\sqrt{2V_1(t)},$$

which is the same as Eq. (B7). If $\alpha=\beta$ in Eq. (B7), then

$$V_2(t)\leqslant e^{-\beta t}V_2(0)+L\sqrt{2V_1(0)}te^{-\beta t}.$$

Finally, if $\alpha\neq\beta$, let $\gamma:=\frac{1}{2}\min(\alpha,\beta)$, then

$$\|(x(t),y(t))-(x_s(t),y_s(t))\|\leqslant\text{const}\times e^{-\gamma t}.$$

If $\alpha=\beta$, then $0<\gamma<\frac{1}{2}\alpha$ will do. This shows exponential synchronization of Eqs. (2a) and (2b).

APPENDIX C

Step 1 [*Uniform boundedness of (q,i)*]. Note that

$$F(x,y;u(t))=i-g(f(q)), \tag{C1}$$

$$G(x,y;u(t))=-\frac{1}{L}f(q)-\frac{R}{L}i+\frac{1}{L}e(t) \tag{C2}$$

with $x=q$, $y=i$, and $u(t)=e(t)$. Define

$$E(q,i):=\int_{q(0)}^{q}f(u)du+\frac{1}{2}Li^2, \tag{C3}$$

which is the energy stored in the capacitor and the inductor.

Since $-f$ is assumed to be decreasing, f is increasing:

$$(q-q')[f(q)-f(q')]\geqslant\alpha(q-q')^2,$$

so that

$$\lim_{\|(q,i)\|\to\infty}E(q,i)=\infty.$$

Since $-g$ is also decreasing, one sees that

$$A:=\left\{(q,i)\middle|f(q)g(f(q))\leqslant\frac{B^2}{4R},\quad|i|\leqslant\frac{B}{R}\right\}$$

is bounded or empty. We will show that

$$\frac{dE(q,i)}{dt}<0,\quad(q,i)\in\mathbb{R}^2-A. \tag{C4}$$

Note that

$$\frac{dE(q,i)}{dt}=f(q)\frac{dq}{dt}+Li\frac{di}{dt}$$
$$=f(q)[-g(f(q))+i]-Ri^2-if(q)+ie(t)$$
$$=-[f(q)g(f(q))+Ri^2]+ie(t). \tag{C5}$$

The first term represents the power dissipated by the resistive part of the circuit; the second term represents the power supplied by the voltage source. Since

$$\min_{|i|}(R|i|^2 - B|i|) = -\frac{B^2}{4R},$$

one sees that, for an arbitrary i,

$$\frac{dE(q,i)}{dt} \leqslant -f(q)g(f(q)) + \frac{B^2}{4R}. \tag{C6}$$

Therefore, for q satisfying

$$f(q)g(f(q)) > \frac{B^2}{4R},$$

the right side of Eq. (C6) is negative, and hence Eq. (C4) follows, so that $(q(t),i(t))$ is uniformly bounded.

Step 2 [*uniform boundedness of* (q_s,i_s)]. Given a uniformly bounded i of Eqs. (1), one must show that (q_s,i_s) is also uniformly bounded. Let $E_s(q_s,i_s)$ be the energy defined in a manner similar to Eq. (C3):

$$E_s(q_s,i_s) := \int_{q_s(0)}^{q_s} f(u)du + \frac{1}{2}Li_s^2. \tag{C7}$$

Then

$$\begin{aligned}\frac{dE_s(q_s,i_s)}{dt} &= f(q_s)[-g(f(q_s)) + i(t)]\\ &\quad -Ri_s^2 - i_s f(q_s) + i_s e(t)\\ &= -f(q_s)[g(f(q_s)) + i_s - i(t)] - Ri_s^2 + i_s e(t)\\ &:= -P(q_s,i_s).\end{aligned} \tag{C8}$$

Since the first subsystem of Eqs. (2) is driven by i, the term $i_s - i$ in Eq. (C8) does not cancel, as it does in Eq. (C5). Note, however, that

$$P(q_s,i_s) \leqslant f(q_s)g(f(q_s)) + i_s f(q_s) + I|f(q_s)| + Ri_s^2 + B|i_s|,$$

where I is a bound on $|i(t)|$. This implies that

$$\lim_{\|q_s,i_s\|\to\infty} P(q_s,i_s) = \infty$$

and hence

$$A_s := \{(q_s,i_s) | P(q_s,i_s) \geqslant 0\}$$

is bounded or empty. Therefore,

$$\frac{dE_s(q_s,i_s)}{dt} < 0, \quad (q_s,i_s) \in \mathbb{R}^2 - A_s,$$

which, together with

$$\lim_{\|(q_s,i_s)\to\infty\|} E_s(q_s,i_s) = \infty,$$

yields a uniform boundedness of (q_s,i_s).

Step 3 (exponential synchronization). In order to apply proposition 2.1, note that [compare Eqs. (1) and (2) with Eqs. (23)–(26)]

$$F(x,y;u(t)) = i - g(f(q)),$$

$$G(x,y;u(t)) = -\frac{1}{L}f(q) - \frac{R}{L}i + \frac{1}{L}e(t),$$

with $x = q$, $y = i$, and $u(t) = e(t)$.

For $i = y$ fixed, $F(x,y;u(t)) = i - g(f(q))$ is decreasing with respect to $x = q$, since $-g$ and $-f$ are both decreasing. For $x = q$ fixed, however,

$$G(x,y;u(t)) = -\frac{1}{L}f(q) - \frac{R}{L}i + \frac{1}{L}e(t)$$

is decreasing with respect to i, because it is linear with a negative coefficient. $G(f,y;u(t))$ is Lipschitz with respect to x because f is Lipschitz. Therefore, Eqs. (25) and (26) exponentially synchronize with (23) and (24).

[1] L. M. Pecora and T. L. Carroll, Phys. Rev. Lett. **64**, 821 (1990).
[2] L. M. Pecora and T. L. Carroll, Phys. Rev. A **44**, 2374 (1991).
[3] T. L. Carroll and L. M. Pecora, Physica D **67**, 126 (1993).
[4] T. L. Carroll and L. M. Pecora, IEEE Trans. Circuits Syst. **40**, 646 (1993).
[5] R. He and P. G. Vaidya, Phys. Rev. A **46**, 7387 (1992).
[6] O. E. Rössler, Ann. (N.Y.) Acad. Sci. **316**, 376 (1979).
[7] J. J. Hopfield, Proc. Natl. Acad. Sci. USA **81**, 3088 (1984).
[8] T. Matsumoto, L. O. Chua, and S. Tanaka, Phys. Rev. A **30**, 1155 (1984).
[9] P. S. Linsay, Phys. Rev. Lett. **47**, 1349 (1981).
[10] T. Matsumoto, L. O. Chua, and S. Tanaka, Phys. Rev. A **30**, 1155 (1984).
[11] S. Tanaka, T. Matsumoto, and L. O. Chua, Physica D **28**, 317 (1987).
[12] S. Tanaka, T. Matsumoto, J. Noguchi, and L. O. Chua, Phys. Lett. **157A**, 37 (1991).
[13] S. Tanaka, J. Noguchi, S. Higuchi, and T. Matsumoto, IEICE Trans. Electron. **E-74**, 1406 (1991).
[14] S. Higuchi, S. Tanaka, M. Komuro, and T. Matsumoto, in *Advanced Series in Dynamical Systems*, edited by H. Kawakami (World Scientific, Singapore, 1991), Vol. 10, pp. 119–138.
[15] S. Tanaka, T. Matsumoto, and L. O. Chua, in *Proceedings of the IEEE International Symposium on Circuit and Systems* (IEEE, New York, 1985).
[16] S. Tanaka, S. Higuchi, and Matsumoto, Phys. Rev. E **54**, 6014 (1996).
[17] T. Matsumoto, M. Komuro, H. Kokubu, and R. Tokunaga, *Bifurcations* (Springer-Verlag, Tokyo, 1993).
[18] P. S. Linsay, Phys. Rev. Lett. **47**, 1349 (1981).

[19] J. Testa, J. Perez, and C. Jeffries, Phys. Rev. Lett. **48**, 714 (1982).
[20] R. W. Rollins and E. R. Hunt, Phys. Rev. Lett. **49**, 1295 (1982).
[21] S. D. Bronson, D. Dewey, and P. S. Linsay, Phys. Rev. A **28**, 1201 (1983).
[22] H. Ikezi, J. S. deGrassie, and T. H. Jenson, Phys. Rev. A **28**, 1207 (1983).
[23] J. Cascais, R. Dilao, and A. Norondacosta, Phys. Lett. **93A**, 213 (1983).
[24] E. R. Hunt and R. W. Rollins, Phys. Rev. A **29**, 1000 (1984).
[25] T. Klinker, W. M. Ilse, and W. Lauterborn, Phys. Lett. **101A**, 371 (1984).
[26] M. F. Bocko, D. H. Douglass, and H. Frutchy, Phys. Lett. **104A**, 388 (1984).
[27] J. M. Perez, Phys. Rev. A **32**, 2513 (1985).
[28] J. Mevissen, R. Seal, and L. Waters, Phys. Rev. A **32**, 2990 (1985).
[29] T. Matsumoto and M. Nishi, in *Proceedings of IUTAM Chaos*, edited by F. Moon (Kluwer Academic, Boston, 1998).
[30] I. Balberg and H. Arbell, Phys. Rev. E **49**, 110 (1984).
[31] D. J. Gauthier and J. C. Biefang, Phys. Rev. Lett. **77**, 1751 (1996).
[32] T. C. Newell, P. M. Alsing, A. Gavrielides, and V. Kovanis, Phys. Rev. Lett. **72**, 1647 (1994).
[33] M. Kuroda, Y. Matsui, and M. Nakai, Trans. Jpn. Soc. Mech. Eng., Ser. C **61**, 808 (1995).
[34] J. M. T. Thompson, Proc. R. Soc. London, Ser. A **387**, 407 (1983).
[35] K. M. Cuomo and A. V. Oppenheim, Phys. Rev. Lett. **71**, 65 (1993).
[36] K. M. Cuomo, A. V. Oppenheim, and S. H. Strogatz, Int. J. Bifurcation Chaos Appl. Sci. Eng. **3**, 1629 (1993).
[37] S. H. Strogatz, *Nonlinear Dynamics and Chaos* (Addison-Wesley, Reading, MA, 1994).
[38] L. Markus and H. Yamabe, Osaka Math. J. **12**, 305 (1960).
[39] J. Szarski, *Differential Inequalities* (Polish Scientific, Warszawa, 1967).